Exercise 39

Show that f is continuous on $(-\infty, \infty)$.

$$f(x) = \begin{cases} 1 - x^2 & \text{if } x \leq 1 \\ \ln x & \text{if } x > 1 \end{cases}$$

Solution

The function is continuous on $(-\infty, 1)$ because, assuming $a < 1$,

$$\begin{aligned} \lim_{x \to a} f(x) &= \lim_{x \to a} (1 - x^2) \\ &= \lim_{x \to a} 1 - \lim_{x \to a} x^2 \\ &= 1 - \left(\lim_{x \to a} x \right) \left(\lim_{x \to a} x \right) \\ &= 1 - (a)(a) \\ &= 1 - a^2 \\ &= f(a). \end{aligned}$$

The function is continuous on $(1, \infty)$ because, assuming $a > 1$,

$$\begin{aligned} \lim_{x \to a} f(x) = \lim_{x \to a} \ln x &= \ln \left(\lim_{x \to a} x \right) \\ &= \ln a \\ &= f(a). \end{aligned}$$

The function is continuous at $x = 1$ because

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1) = 0.$$

Therefore, f is continuous on $(-\infty, \infty)$.